INTRODUCTION to TENSORS

Ref: Arfken, Weber & Harris

1. VECTORS

Def $\vec{x} = (x_1, x_2, x_3)$

$$= x_1 \hat{x} + x_2 \hat{y} + x_3 \hat{z}$$

$$= \sum_{i=1}^{3} x_i \hat{e}_i$$

SCALAR PRODUCT $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$

$$= \sum_{i=1}^{3} x_i y_i \equiv x_i y_i$$

$$= \sum_{i=1}^{3} x_i y_j \delta_{ij} \equiv x_i y_j \delta_{ij}$$

CONVENTION: Sum over repeated indices

⚠ $\vec{x} \cdot \vec{y} \equiv x_i y_i$

$$(\vec{x} \cdot \vec{y})^2 \equiv x_i y_i x_j y_j \neq (x_i y_i)^2$$

$$\left(\sum_{i=1}^{3} x_i y_i\right)^2 \neq \sum_{i=1}^{3} (x_i y_i)^2$$

Would miss out the "mixed" products

CROSS PRODUCT

$$\vec{a} \wedge \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$(a_i \hat{e}_i) \wedge (b_j \hat{e}_j) = (a_i b_i) \underbrace{\hat{e}_i \wedge \hat{e}_j}$$

WHAT is this?

$\hat{x} \wedge \hat{y} = \hat{z}$

$\hat{y} \wedge \hat{x} = -\hat{z}$

$\Leftarrow$ Completely antisymmetric!

$$\hat{e}_i \wedge \hat{e}_j = \varepsilon_{ijk} \hat{e}_k$$

$$\Downarrow$$

$$\vec{a} \wedge \vec{b} = \varepsilon_{ijk} a_i b_j \hat{e}_k \quad \text{or} \quad (\vec{a} \wedge \vec{b})_k = \varepsilon_{ijk} a_i b_j$$

2. TENSORS

Want to encode linear transformations in most convenient way

$\downarrow$

Useful! In electrodynamics everything is linear!

Maths

n-tensor

$T_{i_1 \dots i_n}$ takes some vectors $(k \leq n)$ and gives back a $(n-k)$-tensor

Maybe better to say

Take 1 vector in input and give back a $(n-1)$-tensor

Thanks to SCALAR PRODUCT there is a canonical way of building up linear maps with more & more variables

E.g: 1. Vectors are 1-tensors

$$\vec{v} = v_i \hat{e}_i \dashrightarrow \hat{v}(\vec{w}) = v_i \delta_{ij} v_j$$

2. Rotations are 2-tensors!

$$R\vec{v} = \vec{v}' \Rightarrow v_i' = \sum_j R_{ij} v_j$$

$$\equiv R_{ij} v_j \longleftarrow$$ DO CHECK HOW MANY FREE INDICES YOU HAVE! (~ dimensional analysis)

3. Scalars are 0-tensors!

Why are they useful?

Again encode linear maps of any form in the most general way

Why the formalism?

Avoid cluttering

E.g: Find linear transformations that preserve distances

[what are they?]

$$(\vec{x}_a' - \vec{x}_b')^2 = \delta_{ij} (x_i'^{(a)} - x_i'^{(b)})(x_j'^{(a)} - x_j'^{(b)})$$

HERE INDEX identifies vector not to be confused!

(I could write $\vec{x}'$ & $\vec{y}'$ but want to be instructive)

$$= \underbrace{\delta_{ij} R_{ie} R_{jm}}_{\delta_{em}} (x_e^{(a)} - x_e^{(b)})(x_m^{(a)} - x_m^{(b)})$$

$$\Downarrow$$

$$\sum_{ij} \delta_{ij} R_{ie} R_{jm} =$$

$$\sum R_{i\cdot} R_{i\cdot}$$

$$= \sum_i R_{ie} R_{im}$$

$$= \sum_i (R^t)_{ei} R_{im}$$

$$= (R^t \cdot R)_{em} = \delta_{em} = (\mathbb{1})_{em}$$

All and only orthogonal transformation preserve the norm! $\longleftarrow$ $\boxed{R^t \cdot R = \mathbb{1}}$

$\downarrow$

$$O(d) \supseteq SO(d) \leftarrow \text{-- --} \quad \det R = +1$$

$$SO(d) \oplus \text{PARITY}$$

How do they transform?

$\boxed{\text{ROTATION}}$ $\longrightarrow$ ALL INDICES ROTATE INDIPENDENTLY!

VECTORS $\quad \vec{v} = v_i \hat{e}_i$

New reference system $\quad \hat{e}_i' = R_{ij} \hat{e}_j \left(:= \sum_j R_{ij} \hat{e}_j \right)$

$$\Longrightarrow \vec{v} = v_i \hat{e}_i = v_i (R^{-1})_{ij} \hat{e}_j' = \underbrace{R_{ji} v_i}_{v_j'} \hat{e}_j'$$

TENSORS $\left[\text{E.g. QUADROPOLE: } Q = \begin{pmatrix} x_1^2 & x_1 x_2 & \\ \dots & x_2^2 & \dots \\ & x_2 x_3 & x_3^2 \end{pmatrix} \right]$

$$\overleftrightarrow{T} = T_{i_1 \dots i_m} \hat{e}_{i_1} \wedge \dots \wedge \hat{e}_{i_m} \left(:= \sum_{i_1 = 1}^{d} \sum_{i_2 = 1}^{d} \dots \sum_{i_m = 1}^{d} T_{i_1 \dots i_m} \hat{e}_{i_1} \otimes \dots \otimes \hat{e}_{i_m} \right)$$

$$= T_{i_1 \dots i_m} R_{i_1 j_1} \hat{e}_{j_1}' \otimes \dots \otimes R_{i_m j_m} \hat{e}_{j_m}' =$$

$$= \left(T_{i_1 \dots i_m} R_{i_1 j_1} \dots R_{i_m j_m} \right) \hat{e}_{j_1}' \otimes \dots \otimes \hat{e}_{j_m}'$$

$$\underbrace{\hspace{6cm}}_{T_{j_1 \dots j_m}}$$

$\boxed{\text{PARITY}}$

P : "TRUE VECTORS" $\longrightarrow$ $-$ "TRUE VECTORS"

Practical definition

$P : \vec{x} \longrightarrow -\vec{x}$

↑ POSITION VECTOR

↑

ULTIMATELY, you need to go back to it to learn how an object transforms!

⚠ THE NUMBER of INDICES IS NOT ENOUGH to KNOW how a TENSOR TRANSFORMS UNDER PARITY!

$\boxed{\text{ILLUSTRIOUS EXAMPLES}}$

$\Phi \sim \frac{q}{r} \xrightarrow{P} \Phi \Rightarrow \Phi$ IS A "TRUE" SCALAR

$\boxed{\vec{E}}$: $E_i \sim \partial_i \Phi$

$\partial_i \xrightarrow{P} -\partial_i$

$\Downarrow$

$E_i \xrightarrow{P} -E_i$ $\vec{E}$ IS A "TRUE" VECTOR

$\boxed{\vec{B}}$ $B_i \sim \varepsilon_{ijk} \partial_j A_k$

$\vec{A} \propto \vec{J} \sim q\vec{v} \xrightarrow{P} -\vec{A}$

$\vec{\nabla} \xrightarrow{P} -\vec{\nabla}$

$\vec{v}, \vec{J}, \vec{A}$ & $\vec{\nabla}$ ARE "TRUE" VECTORS

$\Downarrow$

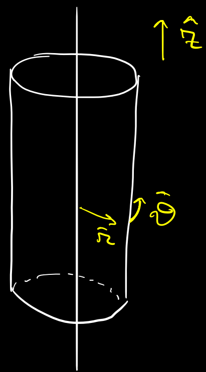
$\vec{B} \xrightarrow{P} \vec{B}$

$\Downarrow$

$\vec{B}$ IS A "PSEUDO"-VECT.

Other example

GRADIENT in CYLINDRICAL COORDINATES



$$\vec{\nabla}_{cart} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \qquad \begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned}$$

$$\vec{\nabla}_{cyl} = \text{Jacob.} \cdot \vec{\nabla}_{cart}$$

$$T : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow \begin{pmatrix} y_1 = \sqrt{x_1^2 + x_2^2} \\ y_2 = \arctan x_2/x_1 \\ y_3 = x_3 \end{pmatrix}$$

$$T^{-1} : \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 = y_1 \cdot \cos y_2 \\ x_2 = y_1 \operatorname{sen} y_2 \\ x_3 = y_3 \end{pmatrix}$$

$$\vec{\nabla} = (\vec{\nabla}_{cart})_i \, \hat{x}_i = (\vec{\nabla}_{cyl})_j \, \hat{y}_j$$

$$= \hat{x}_i \cdot \frac{\partial}{\partial x_i} = \hat{x}_i \cdot \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}$$

$$= \underbrace{T^{-1}_{ik}} \underbrace{\frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}} \cdot \hat{y}_k$$

WE NEED TO COMPUTE BOTH PIECES

E.g: • $k = 3 \qquad \hat{y}_3 = \hat{z}$

* $T_{3i} = ? \longleftarrow \hat{y}_k = T_{ki} \hat{x}_i$

$$\hat{z} = T_{3i} \hat{x}_i \Rightarrow T_{3i} = \delta_{3i}$$

* $$\frac{\partial y_j}{\partial z} \cdot \frac{\partial}{\partial y_j} = \delta_{j3} \frac{\partial}{\partial y_j} = \frac{\partial}{\partial z}$$

• $k = 1$

* $T_{1i} = ?$ $\qquad \hat{y}_1 = \hat{r} = \frac{x_1}{r}\hat{x}_1 + \frac{x_2}{r}\hat{x}_2$

$$T_{11} = \frac{x_1}{r} \qquad T_{12} = \frac{x_2}{r} \qquad T_{13} = 0$$

* $$\frac{\partial y_1}{\partial x_1} \frac{\partial}{\partial y_1} = \frac{x_1}{r} \cdot \frac{\partial}{\partial r} - \frac{x_2}{r^2} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial x_1} \arctan \frac{x_2}{x_1} = - \frac{x_2}{x_1^2} \cdot \frac{1}{1 + x_2^2/x_1^2} = - \frac{x_2}{x_1^2 + x_2^2}$$

* $$\frac{\partial y_1}{\partial x_2} \circ \frac{\partial}{\partial y_1} = \frac{x_2}{r} \frac{\partial}{\partial r} + \frac{x_1}{r^2} \frac{\partial}{\partial \theta}$$

$$(\nabla_{cyl})_1 = \left(\frac{x_1^2}{r^2} + \frac{x_2^2}{r^2}\right) \frac{\partial}{\partial r} + \cancelto{0}{\frac{x_1 x_2 - x_2 x_1}{r^3}} \frac{\partial}{\partial \theta}$$

• $k = 2$

* $T_{2i} = ?$ $\rightsquigarrow$ Finish as exercise